A More General Approach to the Filter Sharpening Technique of Kaiser and Hamming

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Abstract— **Over the years several authors have described a simple technique for filter sharpening. We present a simple explanation of how filter sharpening works and when it will not work. We show that filter sharpening is more broadly applicable than previously noted. We also show that even when filter sharpening does not work in the accepted sense, it still improves the filter's performance.**

Keywords— **Filter Sharpening, IIR Filters, Symmetric FIR Filters.**

I. Introduction

In the late seventies Kaiser and Hamming proposed a method for improving filter performance [3], [2]. Recently the technique was described again [1]. We consider filter sharpening and give a simple explanation of *why* the technique does not always improve a filter's performance. We also show that sharpening a filter twice can overcome some of the method's limitations.

Suppose that one is given a filter whose transfer function is $G(z)$. Assume that in the passband the frequency response is supposed to be 1 and in the stopband the frequency response is supposed to be 0. Suppose that one has a filter whose frequency response is moderately close to one throughout the filter's passband and is moderately close to zero throughout the filter's stopband. Suppose one needs a better filter.

If one had a function, $F(z)$, that took numbers that were near zero and made them closer to zero that would be a step in the right direction. One would consider the filter defined by $F(G(z))$ and one would know that its frequency response was better than the frequency response of $G(z)$.

The most obvious choice for $F(z)$ is z^2. This function takes terms of order ϵ (where $|\epsilon| << 1$) and converts them into terms of order ϵ^2. The function $1+(z-1)^2$ performs a similar "service" for filters that are near 1 in their passband. The problem with these functions is that the first function improves performance in the stopband but hurts performance in the passband. The second function helps performance in the passband but hurts it in the stopband.

We would like to find a function that helps us in both the passband and the stopband. What is needed is a function, $F(z)$, that satisfies four conditions:
1. F(0) = 0.
2. F'(0) = 0.
3. F(1) = 1.
4. F'(1) = 0.

These conditions force the function to behave like az^2 near zero *and* like $1+b(z-1)^2$ near one. If one would like a polynomial to satisfy these four conditions then the polynomial must, at the very least, be cubic—any lower order polynomial will not have enough coefficients to allows us to meet all of the conditions.

Consider the general cubic polynomial:

$$F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3.$$

The four conditions on $F(z)$ lead to four conditions on the coefficients:

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1. $F(0) = a_0 = 0$.
2. $F'(0) = a_1 = 0$.
3. $F(1) = a_2 + a_3 = 1$.
4. $F'(1) = 2a_2 + 3a_3 = 0$.

Solving for a_2 and a_3, we find that $a_2 = 3$ and $a_3 = -2$. Thus, one function that satisfies all the criteria is:

$$F(z) = 3z^2 - 2z^3.$$

It is easy to see that any number near zero is made smaller by this function. Rewriting $F(z)$ as:

$$F(z) = 1 - 3(z-1)^2 - 2(z-1)^3 \quad (1)$$

we find that any number near one is made nearer to one by $F(z)$. In fact we find that if $z = \epsilon, |\epsilon| << 1$, then after passing z through $F(z)$ we have approximately $3\epsilon^2$, and if $z = 1+\epsilon$ then after passing z through $F(z)$ we have approximately $1 - 3\epsilon^2$. The function $F(z)$ is the function Kaiser and Hamming found in their classic paper [3].

Consider the filter whose transfer function is given by $F(G(z)) = -2G^3(z) + 3G^2(z)$—consider the *sharpened* filter. This filter ought to be a better filter than $G(z)$; $F(G(z))$ should be closer to one in the filter's passband and should be closer to zero in the filter's stopband than $G(z)$ is [3].

II. A More Careful Examination

Let us see why filter sharpening does not always improve filter performance. Suppose that one has a filter whose value at a point, z_0, in the filter's passband is:

$$1 + j\epsilon + (\delta_r + j\delta_i)\epsilon^2, \qquad \epsilon, \delta_r, \delta_i \in \mathcal{R}, \qquad |\epsilon| << 1.$$

Making use of the fact that $\sqrt{1+z} = 1 + z/2 + O(z^2)$ (where by $O(z^n)$ we mean that the term being represented by $O(z^n)$ is less than $C|z^n|$ for some fixed positive C and for all sufficiently small z), we find that the magnitude of the filter response is:

$$\begin{aligned} |G(z)| &= \sqrt{(1+\delta_r\epsilon^2)^2 + (\epsilon + \delta_i\epsilon^2)^2} \\ &= (1+\delta_r\epsilon^2)\sqrt{1 + (\epsilon+\delta_i\epsilon^2)^2/(1+\delta_r\epsilon^2)} \\ &= (1+\delta_r\epsilon^2)(1+\epsilon^2/2) + O(\epsilon^3) \\ &= 1 + \left(\delta_r + \frac{1}{2}\right)\epsilon^2 + O(\epsilon^3). \end{aligned} \quad (2)$$

Even though $j\epsilon$ is of order ϵ, this term's effect on the magnitude of the frequency response is only of order ϵ^2—it is smaller than we expected.

Making use of (1) one finds that after "sharpening" $1 + j\epsilon + O(\epsilon^2)$ one is left with $F(G(z)) = 1 + 3\epsilon^2 + O(\epsilon^3)$. As $1 + 3\epsilon^2$ is real, it is the approximate magnitude of the result as well. From (2) one finds that if $|\delta_r + 1/2| < 3$, the magnitude response is *harmed* by sharpening.

This does not tell the whole story, however. One positive change is made by sharpening—the phase of the frequency response becomes closer to zero. After sharpening once, we find that the transfer function is (except for term of order ϵ^3 or higher) $1 + 3\epsilon^2$, and this value is real number. We see that $F(G(z))$ can profitably be sharpened, and, making use of (1), we find that after sharpening:

$$F(F(G(z))) \approx 1 - 27\epsilon^4.$$

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III. Sharpening Low-pass Filters: General Theory

Suppose that $G(z)$ is the transfer function of a low-pass filter. Assuming that the low-pass filter passes DC signals without change, we find that $G(1) = 1$. Furthermore, let us assume that $G(z)$ is a rational function with real coefficients. Then we know that $G(1) = 1$ and $G'(1), G''(1) \in \mathcal{R}$. Making use of the Taylor expansion of $G(z)$ about $z = 1$ we find that:

$$G(z) = 1 + G'(1)(z-1) + G''(1)(z-1)^2/2 + O((z-1)^3).$$

If we are interested in the frequency response in a neighborhood of $z = 1$, we are interested in the behavior of the transfer function when:

$$z = e^{j\Omega} = 1 + j\Omega - \Omega^2/2 + O(\Omega^3), \qquad |\Omega| << 1.$$

In this region:

$$G(z) \approx G(1 + j\Omega - \Omega^2/2) \approx 1 + G'(1)j\Omega - (G'(1) + G''(1))\Omega^2/2$$

(where only terms of order Ω^3 or higher have been ignored). If $G'(1) \neq 0$, then we find that $G(z)$ is equal to 1 plus a term that is, to leading order, pure imaginary. We have seen that the sharpening process does not work well in such cases—it will not sharpen the filter (though it will make the filter's phase very close to 0^o for small Ω).

Let us consider the filter whose transfer function is $F(G(z))$. From the definition of $F(z)$ it is clear that if $G(z)$ is a rational function with real coefficients then so is $F(G(z))$. Assuming that $G(z)$ is low-pass, $G(1) = 1$. From the definition of $F(z)$, we find that $F(G(1)) = 1$. It is clear that:

$$\frac{d}{dz}F(G(z)) = F'(G(z))G'(z).$$

Evaluating this at $z = 1$ and using that fact that $F'(1) = 0$ we find that the derivative of the new transfer function at $z = 1$ is:

$$\left.\frac{d}{dz}F(G(z))\right|_{z=1} = F'(1)G'(1) = 0 \cdot G'(1) = 0.$$

Thus, for $z = e^{j\Omega}, |\Omega| << 1$ we find that (except for terms of order Ω^3 or higher):

$$F(G(e^{j\Omega})) \approx 1 + 0 \cdot j\Omega + -\left(\left.\frac{d^2}{dz^2}F(G(z))\right|_{z=1}\right)\Omega^2/2.$$

Near $z = 1$, near DC, this term is pure real and, as we have shown, such a filter can profitably be sharpened. We see that the class of all low-pass filters (that have transfer functions that are rational functions with real coefficients) that have been sharpened once can profitably be sharpened again.

IV. Sharpening Low-pass Filters: An Example

Consider, for example,

$$G(z) = \frac{z}{2z-1};$$

this is the transfer function of a simple low-pass filter. The Bode plots corresponding to this function are given by the solid lines in Figure 1. Consider the value of the transfer function for:

$$z = e^{j\Omega} \approx 1 + j\Omega - \Omega^2/2, \qquad 0 < \Omega << 1.$$

As the Taylor series expansion of $G(z)$ about $z = 1$ is:

$$G(z) = 1 - (z-1) + 2(z-1)^2 + O((z-1)^3),$$

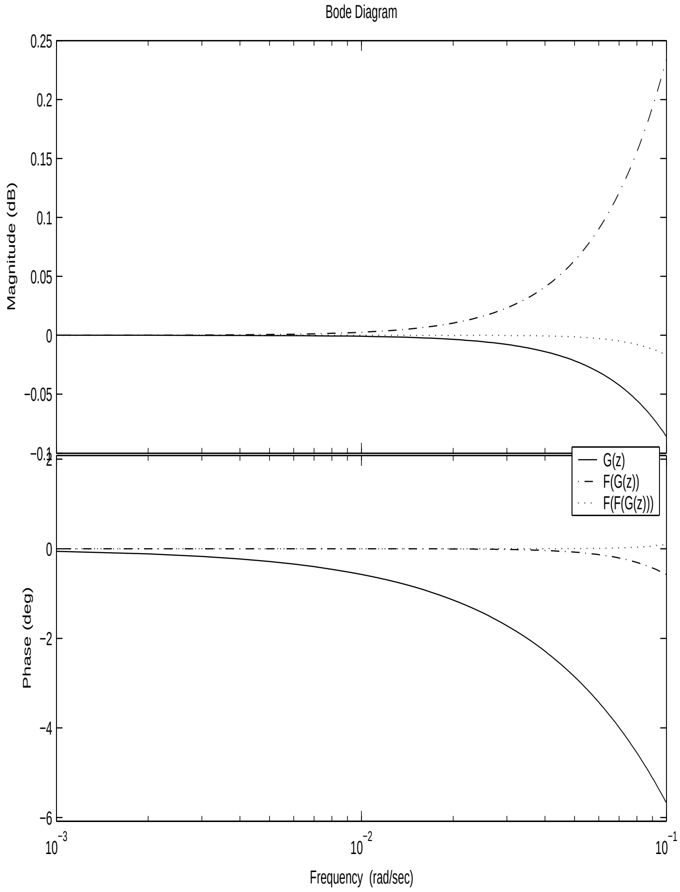

Fig. 1. The Bode Plots of the Three Filters.

we find that for $z \approx 1 + j\Omega - \Omega^2/2$ we have:

$$G(z) = 1 - j\Omega - (3/2)\Omega^2 + O(\Omega^3).$$

From (2) it is clear that $|G(z)| \approx 1 - \Omega^2$. Additionally, because $G(z) \approx 1 - j\Omega - (3/2)\Omega^2$, we find that after sharpening the filter, $F(G(z)) \approx 1 + 3\Omega^2$. Looking at the "dash-dot" line in the Bode plots of Figure 1 we see the deviation of the magnitude response from 1—from 0 dB—has indeed increased. That is, sharpening the filter has made its magnitude response *worse*. Note, however, that the frequency response for small Ω is now very nearly pure real—its phase is nearly zero. If we sharpen the sharpened filter, if we consider $F(F(G(z)))$, we find that $F(F(G(z))) \approx 1 - 27\Omega^4$. (See the dotted lines in Figure 1.) This is a real gain; the frequency response of the twice sharpened filter is much nearer to one and its phase is much nearer to zero.

V. The Classical Use

Filter sharpening is generally used to sharpen symmetric finite impulse response (FIR) filters. (For more information on this use, please see [3].) The transfer function of a symmetric FIR filter is:

$$G(z) = \sum_{k=-N}^{N} a_{|k|}z^{-k}.$$

Clearly:

$$G'(z) = \sum_{k=-N}^{N} (-k)a_{|k|}z^{-(k+1)}$$

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and:

$$G'(1) = \sum_{k=-N}^{N} (-k)a_{|k|} = 0.$$

We find that all low-pass symmetric FIR filters are members of the class of filters for which the sharpening technique is most helpful. The filter whose transfer function is $F(G(z))$ will be more nearly ideal than the original filter. As we saw in sections III and IV, symmetric FIR filters are not the only members of that class; there is even a set of infinite impulse response (IIR) filters that are members of that class.

VI. Conclusions

We have explained why the sharpening procedure of Kaiser and Hamming does not always work. We have shown that even when their procedure does not work to sharpen a filter, it may help by improving the phase response of the filter. We have also seen that sharpening a filter twice can help overcome the problems with the sharpening method. We provide a simple example in which the effects discussed are observed, and we give an example of a class of IIR filters for which the sharpening technique is effective.

References

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