A Very Brief Introduction to Linear Time-Invariant (LTI) Systems

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1 What is a Linear Time-Invariant System?

Linear time-invariant (LTI) systems are systems that are both linear and time-invariant. Let $x_1(t)$ and $x_2(t)$ be any two signals. Suppose that the output of a system to $x_1(t)$ is $y_1(t)$ and the ouptut of the system to $x_2(t)$ is $y_2(t)$. If this *always* implies that the output of the system to $\alpha_1 x_1(t)+\alpha_2 x_2(t)$ is $\alpha_1 y_1(t)+\alpha_2 y_2(t)$, then the system is linear and the *superposition principle* is said to hold. A system is said to be time invariant if when $y(t)$ is the output that corresponds to $x(t)$, then for any τ, $y(t-\tau)$ is the output that corresponds to $x(t-\tau)$.

1.1 Examples

1. An ideal amplifier, a system for which $y(t) = Cx(t)$, is an LTI.
2. A zero-order hold, a system whose output for $kT_s \leq t < (k+1)T_s$ is equal to $x(kT_s)$ is linear but is not time invariant. (Can you see why?)
3. A system for which $y(t) = x^3(t)$ is time invariant but not linear.

2 The Delta Function and the Sifting Property

Consider the "function" $\delta(t)$ defined to be zero for all $t \neq 0$ and to have unit area. This "function" is infinite at $t = 0$ and has a total area of one. If one multiplies a delta function by any other continuous function, $x(t)$, the result will be a new function that is zero whenever $t \neq 0$ and whose area has been multiplied by $x(0)$. That is, $\delta(t)x(t) = x(0)\delta(t)$.

Consider the integral

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.$$

By our previous logic, this is the same as

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\, d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\, d\tau = x(t).$$

That is,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.$$

This is known as the *sifting property* of the delta function. Recalling that the integral is a "generalized sum," we can say that the sifting property says that every function can be written as a "weighted sum" of time-shifted delta functions.

2.1 How to Think About a Delta Function

It often helps to think of a delta function as a function that is zero outside of the region $|t| < \epsilon/2$ and that has height $1/\epsilon$ when $|t| < \epsilon/2$. (Epsilon should be thought of as a *very* small number.)

3 Combining the Sifting Property of the Delta Function and the Linearity and Time-Invariance of a System

Suppose that one has an LTI system and one knows that the systems response to a delta function is the function $h(t)$. Because of its importance, the function $h(t)$ has its own name; it is known as the system's *impulse response*. It is possible to take our knowledge of the system's response to a delta function, of the system's impulse response, and convert it into knowledge of the system's output to any input.

Suppose that the system's input is $x(t)$. Make use of the sifting property of the delta function to write

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.$$

That is, write $x(t)$ as a *weighted sum* of time-shifted delta functions. By linearity and time-invariance, the output of the system must be the same weighted sum of the time shifted output of the system to a delta functions. The output of the system is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$$

Once we know the output of the system to a delta function, we know the output of the system to any function whatsoever. *An LTI system is completely characterized by its impulse response.*

3.1 The Moving Average Filter

Suppose that for a particular system

$$h(t) = \begin{cases} 1/T & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}.$$

We find that

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{t-T}^{t} x(\tau)(1/T)\,d\tau \\ &= \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau. \end{aligned}$$

This filter is known as the *moving average* filter.

3.2 The Filter's Response to $\cos(2\pi ft)$

Suppose that $x(t) = \cos(2\pi ft)$. Then if $f = 0$, we find that $y(t) = T$. Otherwise, we find that

$$\begin{aligned} y(t) &= \int_{t-T}^{t} \cos(2\pi ft)\, dt \\ &= \frac{1}{T}\frac{\sin(2\pi ft)}{2\pi f}\bigg|_{t-T}^{T} \\ &= \frac{1}{T}\frac{\sin(2\pi ft) - \sin(2\pi f(t-T))}{2\pi f} \\ &= \frac{1}{T}\frac{\sin(2\pi ft)(1-\cos(2\pi fT)) + \cos(2\pi ft)\sin(2\pi fT)}{2\pi f}. \end{aligned}$$

As $f \to \infty$, the output tends to zero. When fT is small, the coefficent of the sine tends to zero as $(fT)^2$ and can be treated like zero. What remains is

$$\frac{\cos(2\pi ft)\sin(2\pi fT)}{2\pi fT} \to \cos(2\pi ft).$$

That is, at very low frequencies the output is very similar to the input.

4 The Convolution

When one takes two functions, $x(t)$ and $h(t)$, and one calculates a new function, $y(t)$, by calculating

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau.$$

one is said to be *convolving* the functions $x(t)$ and $h(t)$, and the function $y(t)$ is the *convolution* of $x(t)$ and $h(t)$. The convolution of $x(t)$ and $h(t)$ is denoted by $x * h(t)$. By making use of the substitution $u = t - \tau, du = -d\tau$, we find that

$$\begin{aligned} y(t) &= x * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \\ &= -\int_{\infty}^{-\infty} x(t-u)h(u)\, du \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} h(u)x(t-u)\,du \\
&= h * x(t).
\end{aligned}
$$

That is, *convolution is a commutative operation.*

5 The Frequency Response of an LTI System

We now consider the response of an LTI system to a special class of signals – the sinusoids. First we consider the system's response to $x(t) = \mathrm{e}^{2\pi jft}$. For this input, the output of the system is

$$
y(t) = h*x(t) = \int_{-\infty}^{\infty} h(t)\mathrm{e}^{2\pi jf(t-\tau)}\,d\tau = \mathrm{e}^{2\pi jft}\int_{-\infty}^{\infty} h(t)\mathrm{e}^{-2\pi jf\tau}\,d\tau = \mathrm{e}^{2\pi jft}H(f)
$$

where $H(f)$ is the Fourier transform of the impulse response. Because of its importance, the function $H(f)$ is called the *frequency response* of the system.

Next consider the output of our system to the practically very important cosine function. Let $x(t) = \cos(2\pi ft)$. As

$$
x(t) = \cos(2\pi ft) = \frac{\mathrm{e}^{2\pi jft} + \mathrm{e}^{-2\pi jft}}{2},
$$

the output of the system will be

$$
y(t) = \frac{H(f)\mathrm{e}^{2\pi jft} + H(-f)\mathrm{e}^{-2\pi jft}}{2}.
$$

In most cases of interest, $h(t)$ is a real-valued function. In such cases, we find that

$$
\begin{aligned}
H(-f) &= \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi j(-f)t}h(t)\,dt \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{2\pi jft}h(t)\,dt \\
&= \int_{-\infty}^{\infty} \overline{\mathrm{e}^{-2\pi jft}}h(t)\,dt \\
&\overset{h(t)\in\mathcal{R}}{=} \overline{\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi jft}h(t)\,dt} \\
&= \overline{H(f)}.
\end{aligned}
$$

When $h(t)$ is real-valued, we find that

$$y(t) = \mathrm{Re}(H(f)\mathrm{e}^{2\pi jft}).$$

A simple calculation shows that

$$\begin{aligned} y(t) &= \mathrm{Re}(H(f))\cos(2\pi ft) - \mathrm{Im}(H(f))\sin(2\pi ft) \\ &= |H(f)|\left(\frac{\mathrm{Re}(H(f))}{|H(f)|}\cos(2\pi ft) - \frac{\mathrm{Im}(H(f))}{|H(f)|}\sin(2\pi ft)\right). \end{aligned}$$

Recalling that $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and that if $\alpha, \beta \in \mathcal{R}$ and $\alpha^2 + \beta^2 = 1$, then the equations $\cos(\theta) = \alpha$ and $\sin(\theta) = \beta$ have a simultaneous solution, we find that

$$y(t) = |H(f)|\cos(2\pi ft + \theta)$$

where θ solves

$$\begin{aligned} \cos(\theta) &= \frac{\mathrm{Re}(H(f))}{|H(f)|} \\ \sin(\theta) &= \frac{\mathrm{Im}(H(f))}{|H(f)|}. \end{aligned}$$

Noting that the θ that solves the above equations is precisely $\angle H(f)$, we find that

$$y(t) = |H(f)|\cos(2\pi ft + \angle H(f)).$$

This says that when a cosine (or, by time-invariance, a sine) of frequency f is passed through an LTI system, the cosine is amplified (or a attenuated) by a factor of $|H(f)|$ and it is shifted by $\angle H(f)$ degrees.

5.1 The Frequency Response of the Moving Average Filter

Let

$$h(t) = \begin{cases} 1/T & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}.$$

The frequency response of the system is

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi j f t} h(t)\, dt \\
&= \frac{1}{T}\int_{0}^{T} \mathrm{e}^{-2\pi j f t}\, dt \\
&= \frac{1}{T}\left.\frac{\mathrm{e}^{-2\pi j f t}}{-2\pi j f}\right|_{0}^{T} \\
&= \frac{\mathrm{e}^{-2\pi j f T} - 1}{-2\pi j f T} \\
&= \mathrm{e}^{-\pi j f T}\frac{\mathrm{e}^{-\pi j f T} - \mathrm{e}^{\pi j f T}}{-2\pi j f T} \\
&= \mathrm{e}^{-\pi j f T}\frac{\sin(\pi f T)}{\pi f T} \\
&= \mathrm{e}^{-\pi j f T}\mathrm{sinc}(fT).
\end{aligned}$$

Note that $|H(f)|$ is bounded by $1/(\pi f T)$. As the frequecy of the input increases, the bound on the output decreases. The moving average filter is a kind of low-pass filter.

6 Stability

It is often important to know that the output of a system remains "reasonable" as long as the input to the system remains "reasonable." In particular, if we input a signal that is not too big, we expect that the output will not be too big either.

We say that a system is bounded-input bounded-output (BIBO) stable if whenever the input to the system is bounded so is the output. Let us see what we can say about the output of our system when we know that the input is bounded – that the input satisfies $|x(t)| < C$. When this condition holds, we find that

$$\begin{aligned}
|y(t)| &= |h * x(t)| \\
&= |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau| \\
&\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau
\end{aligned}$$

$$\leq \; C\int_{-\infty}^{\infty} |h(\tau)|\, d\tau.$$

Clearly if the integral of the absolute value of the impulse response is bounded, the system is BIBO stable. It is not hard to show that if the integral is not bounded, then the system is not BIBO stable. We find that *for a system to be BIBO stable it is necessary and sufficient that its impulse response be absolutely integrable.*

6.1 The Stability of the Moving Average Filter

As the integral of the absolute value of the impulse response of the moving average filter is one, it is clear that the filter is stable. Moreover, looking back at the argument we made for stability, we also find that for the moving average filter, $|y(t)| \leq \max |x(t)|$.

7 The RC Low-Pass Filter

Consider the circuit of Figure 1. As we will see, this circuit is a low-pass filter. We start by analyzing the circuit. As $v_{\text{out}}(t)$ is the voltage on a capacitor, $Q(t) = Cv_{\text{out}}(t)$ and $i(t) = Q'(t) = Cv'_{\text{out}}(t)$. As the voltage that drops across the entire circuit is $v_{\text{in}}(t)$, we find that

$$RCv'_{\text{out}}(t) + v_{\text{out}}(t) = v_{\text{in}}(t).$$

Assuming that the capacitor is initially discharged, that $v_{\text{out}}(0^-) = 0$, it is not hard to show (using the method of variation of parameters, for instance) that

$$v_{\text{out}}(t) = \frac{e^{-t/(RC)}}{RC}\int_{0^-}^{t} e^{y/(RC)} v_{\text{in}}(y)\, dy.$$

When $v_{\text{in}}(t) = \delta(t)$, we find that

$$h(t) = v_{\text{out}}(t) = \frac{e^{-t/(RC)}}{RC}, \qquad t \geq 0.$$

Clearly, $h(t) = 0$ until $t = 0$; by assumption, nothing happened until the delta function arrived at $t = 0$. We find that the frequency response of the

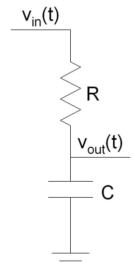

Figure 1: A simple RC low-pass filter

system is

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi j f t} h(t)\, dt \\
&= \int_{0}^{\infty} \mathrm{e}^{-2\pi j f t} \mathrm{e}^{-t/(RC)}/(RC)\, dt \\
&= \frac{1}{RC} \int_{0}^{\infty} \mathrm{e}^{-(2\pi j f + 1/(RC))t}\, dt \\
&= \frac{1}{RC} \left. \frac{\mathrm{e}^{-(2\pi j f + 1/(RC))t}}{-(2\pi j f + 1/(RC))} \right|_{0}^{\infty} \\
&= \frac{1}{RC} \frac{1}{2\pi j f + 1/(RC)} \\
&= \frac{1}{2\pi j f RC + 1}.
\end{aligned}$$

As

$$\left| \frac{1}{2\pi j f RC + 1} \right| = \frac{1}{\sqrt{(2\pi f RC)^2 + 1}}$$

it is clear that this system is a low-pass filter.

8 Exercises

1. Find the frequency response of the filter whose impulse response, h(t), is given by

$$h(t) = \begin{cases} 1 & |t-2| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

How does this function differ from the frequency response of the moving average filter?

2. Let

$$y(t) = \begin{cases} 1 & x(t) > 0 \\ 0 & x(t) = 0 \\ -1 & x(t) < 0 \end{cases}.$$

Is the system thus defined linear? Is it time-invariant? Explain.

3. Let

$$h(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Plot $y(t)$ when $x(t)$ is:

(a) $x(t) = 1$,

(b) $x(t) = \sin(2\pi \cdot 0.1t)$,

(c) $x(t) = \sin(2\pi t)$,

(d) $x(t) = \sin(2\pi 1.5t)$.

4. Are the LTI systems defined by the following impulse responses stable? Explain.

(a) $h(t) = \mathrm{e}^{-|t|}$,

(b) $h(t) = \mathrm{e}^{-t}$,

(c)

$$h(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

(d)

$$h(t) = \begin{cases} 1/t & t \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

5. Let

$$H(f) = \begin{cases} 1 & |f| < F_1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find $h(t)$.

(b) What kind of filter does $H(f)$ define?