2.6 The Leontief Input-Output Model

Nation's economy in n sectors; $\mathbf{x}$ is **production vector**

Open sector: consumes only; $\mathbf{d}$ is **final demand vector**

Producers create additional **intermediate demands** for own production

Leontief asked: $\exists \mathbf{x} \ni$ produced $\stackrel{?}{=}$ total demand:

$$\left\{ \begin{array}{c} \text{amount} \\ \text{produced} \\ \mathbf{x} \end{array} \right\} = \left\{ \begin{array}{c} \text{intermediate} \\ \text{demand} \end{array} \right\} + \left\{ \begin{array}{c} \text{final} \\ \text{demand} \\ \mathbf{d} \end{array} \right\}$$

Basic assumption: for each sector i $\exists$ **unit consumption vector** $\mathbf{c}_i \in \boldsymbol{R}^n$: inputs needed per unit of output

All units measured in money, e.g.: $M

Purchased	Inputs consumed per unit of output		
from:	Manufacturing	Agriculture	Services
Manufacturing	.5	.4	.2
Agriculture	.2	.3	.1
Services	.1	.1	.3
	$\uparrow$	$\uparrow$	$\uparrow$
	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$

E.g.: What will be consumed by Manufacturing if it produces \$100M worth?

$$100\mathbf{c}_1 = 100\begin{bmatrix} .5 \\ .2 \\ .1 \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \\ 10 \end{bmatrix}$$

or: Manufacturing will demand (or consume) \$50M of Manufacturing, \$20M of Agriculture and \$10M of Services. (Take home: \$20M.)

Intermediate demand:

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 = C\mathbf{x}$$

where **consumption matrix**:

$$C = \begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{bmatrix}$$

Leontief's Input-Output Model:

$$\underset{\text{amount produced}}{\mathbf{x}} = \underset{\text{intermediate demand}}{C\mathbf{x}} + \underset{\text{final demand}}{\mathbf{d}}$$

Rewriting:

$$I\mathbf{x} - C\mathbf{x} = \mathbf{d}$$

or:

$$(I - C)\mathbf{x} = \mathbf{d}$$

E.g., $d = [50\ 30\ 20]^T$, ($50M, $30M, $20M, resp.), we have:

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{bmatrix} = \begin{bmatrix} .5 & -.4 & -.2 \\ -.2 & .7 & -.1 \\ -.1 & -.1 & .7 \end{bmatrix}$$

Solving:

$$\begin{bmatrix} .5 & -.4 & -.2 & 50 \\ -.2 & .7 & -.1 & 30 \\ -.1 & -.1 & .7 & 20 \end{bmatrix} \sim \cdots \approxeq \begin{bmatrix} 1 & 0 & 0 & 226 \\ 0 & 1 & 0 & 119 \\ 0 & 0 & 1 & 78 \end{bmatrix}$$

Therefore, Manufacturing must produce $226M worth, Agriculture $119M worth and Services $78M worth. (!!)

If $(I - C)$ invertible, then:

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

In general:

1) $(I - C)$ is invertible

2) $\mathbf{x}$ is feasible, i.e., $\forall i : x_i \geq 0$

3) C's column sums < 1 (reasonable; otherwise: deficit)

Theorem: If [1]$\forall i : d_i \geq 0$, [2]$\forall i, j : C_{ij} \geq 0$ and [3]column sums < 1, then $\exists\ (I - C)^{-1}$ and $\mathbf{x}$ unique with $\forall i : x_i \geq 0$.

A Formula for $(I - C)^{-1}$

Initially, only final demand $\mathbf{d} \Rightarrow$ intermediate demand $C\mathbf{d}$

	demand that must be met	inputs needed to meet this demand
final demand	$\mathbf{d}$	$C\mathbf{d}$
intermediate demand		
1st round	$C\mathbf{d}$	$C(C\mathbf{d}) = C^2\mathbf{d}$
2nd round	$C^2\mathbf{d}$	$C(C^2\mathbf{d}) = C^3\mathbf{d}$
3rd round	$C^3\mathbf{d}$	$C(C^3\mathbf{d}) = C^4\mathbf{d}$
$\vdots$	$\vdots$	$\vdots$

Production vector $\mathbf{x}$ to meet this demand:

$$\begin{aligned}\mathbf{x} &= \mathbf{d} + C\mathbf{d} + C^2\mathbf{d} + C^3\mathbf{d} + \cdots \\ &= \left(I + C + C^2 + C^3 + \cdots\right)\mathbf{d}\end{aligned}$$

Note the identity (multiply out to see):

$$(I - C)\left(I + C + C^2 + \cdots + C^m\right) = I - C^{m+1}$$

With above assumptions, $C^m \to 0$ and:

$$(I - C)^{-1} \approx I + C + C^2 + C^3 + \cdots + C^m$$

Think of scalar analogy: $C = \frac{1}{2}$

We have for $(I - C)^{-1}$

1) generally, only small m is necessary
2) column j are increase productions due to one unit increase in the final demand from sector j
3) $A\mathbf{x} = \mathbf{b} \rightarrow (I - C)\mathbf{x} = \mathbf{b}$ with $C = I - A$; if A sparse and column sums < 1 then $C^m \rightarrow 0$, etc.