1.9 The Matrix of a Linear Transformation (cont.)

Example: Find the standard matrix A of the linear transformation which rotates each point in $\boldsymbol{R}^2$ counterclockwise about the origin by the angle ϕ.

Solution: $\mathbf{e}_1$ is rotated to $\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, and $\mathbf{e}_2$ to $\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$, so we have:

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

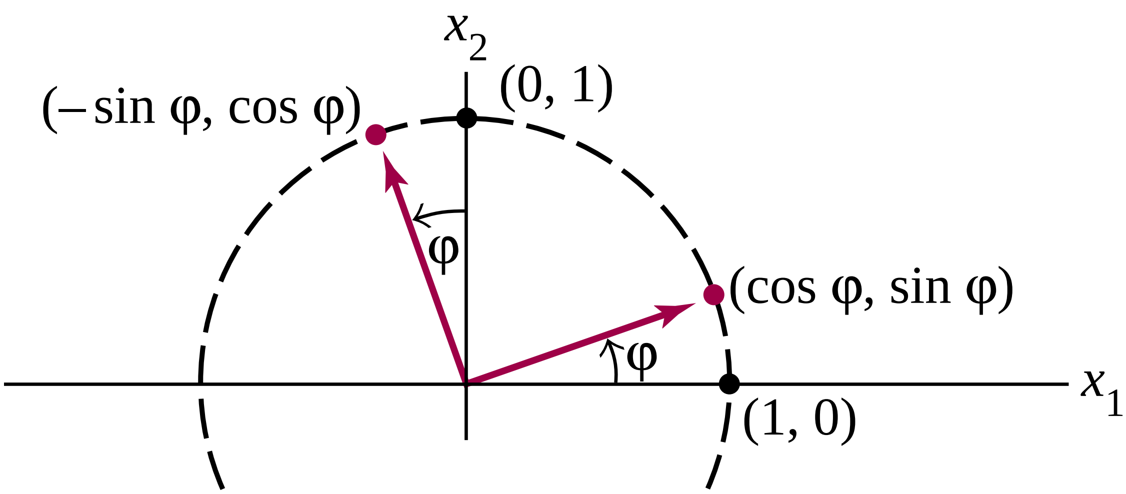


FIGURE 1 A rotation transformation.

Geometric Linear Transformations of $\boldsymbol{R}^2$

Linear transformations of $\boldsymbol{R}^2$ are completely determined by what they do to the columns of I_2

Here, showing effect on entire unit square

Note: one can compose more than one transform $\Rightarrow$
Still, determine final locations of $\mathbf{e}_1$ and $\mathbf{e}_2$'s images

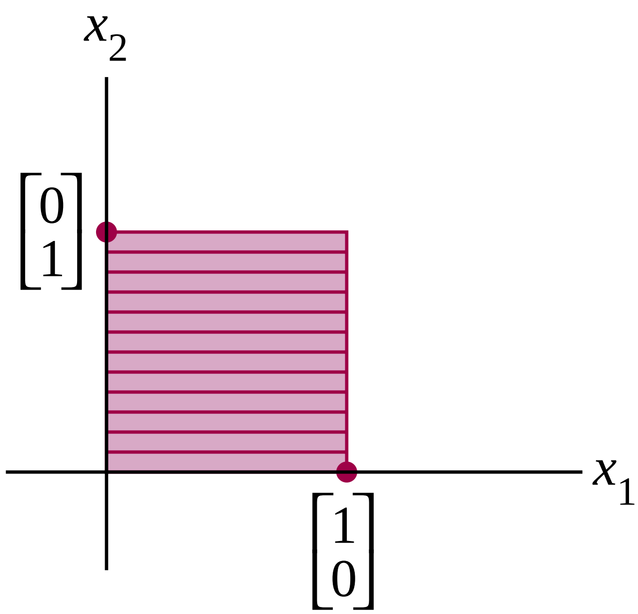
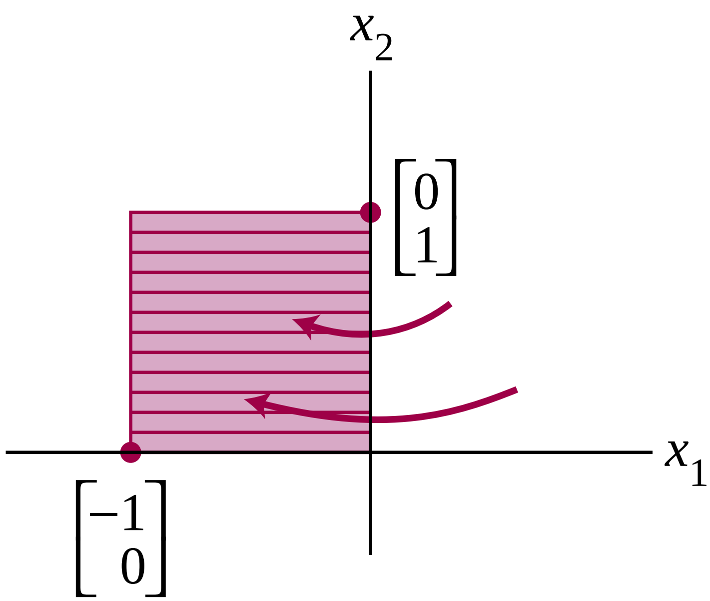
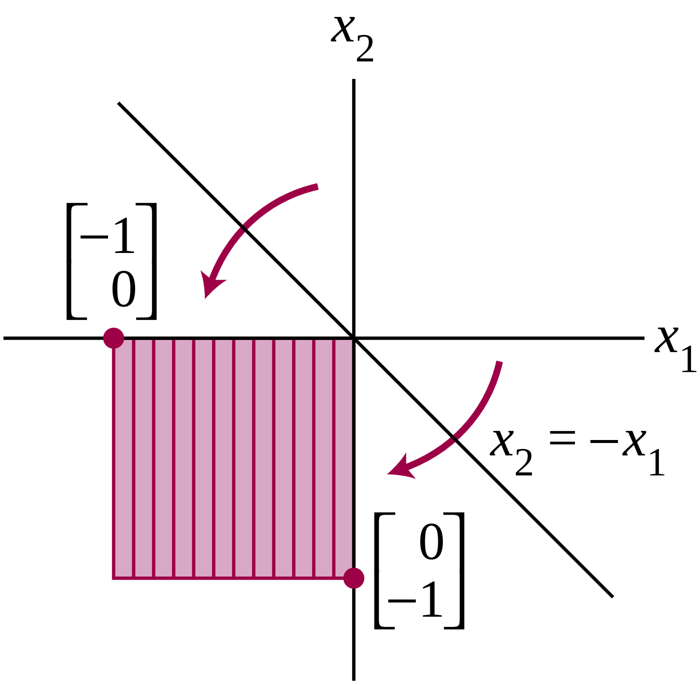


FIGURE 2 The unit square.

Reflection in the x_2-axis

Image of the Unit Square	Standard Matrix
	

Reflection in the line $x_2 = -x_1$

Image of the Unit Square	Standard Matrix
	$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

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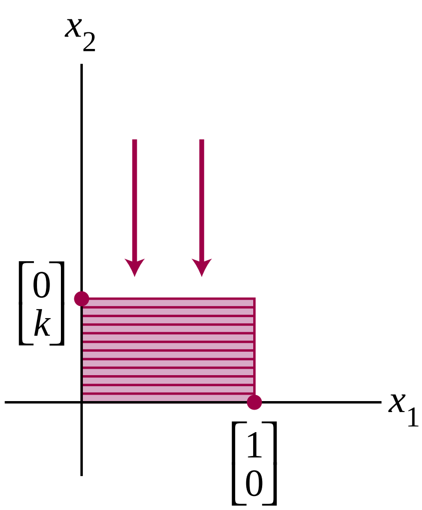
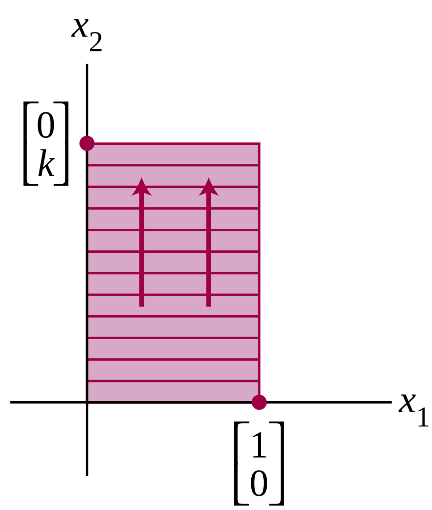
Vertical Contraction and Expansion

Image of the Unit Square	Standard Matrix



$0 < k < 1$



$k > 1$

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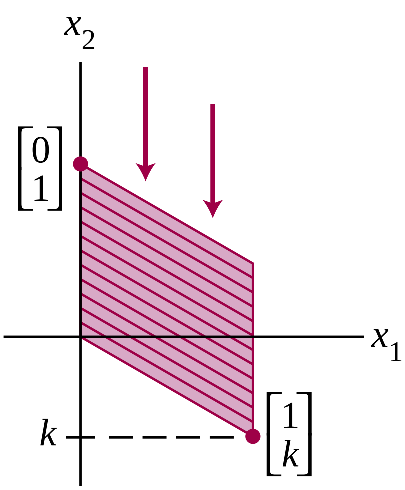
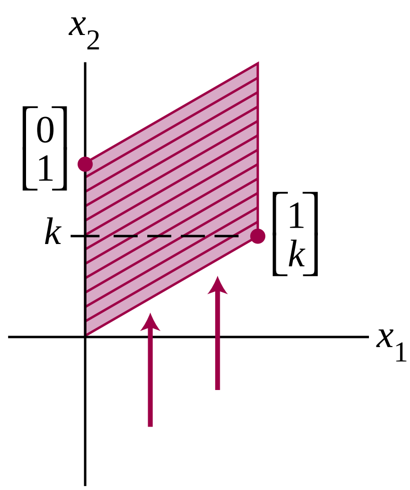
Vertical Shear

Image of the Unit Square	Standard Matrix



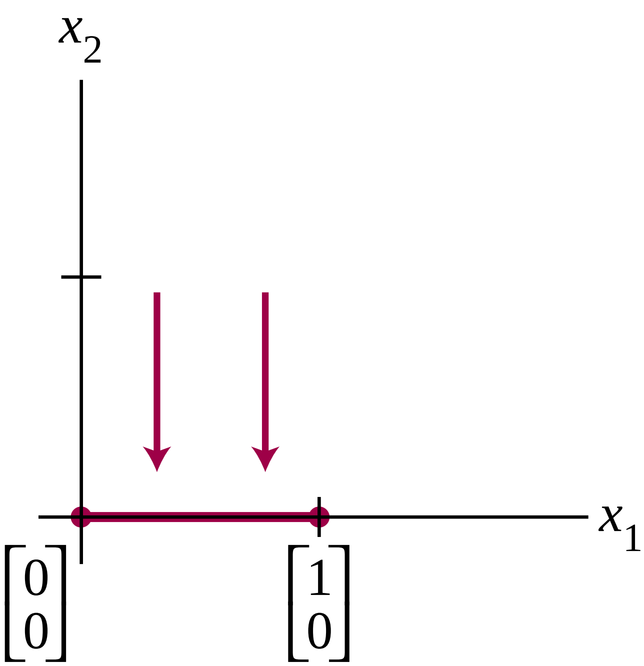
$k < 0$



$k > 0$

$$\begin{bmatrix} & \\ & \end{bmatrix}$$

Projection onto the x_1-axis

Image of the Unit Square	Standard Matrix
	$\begin{bmatrix} & \\ & \end{bmatrix}$

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Definition: Mapping $T : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^m$ is **onto** $\boldsymbol{R}^m$ if $\forall \mathbf{b} \in \boldsymbol{R}^m$, $\exists (\geq 1)\mathbf{x} \in \boldsymbol{R}^n \ni T(\mathbf{x}) = \mathbf{b}$. (I.e., when range equals codomain.)

Onto is existence question

Definition: Mapping $T : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^m$ is **one-to-one** if $\forall \mathbf{b} \in \boldsymbol{R}^m$, $\exists (\leq 1)\mathbf{x} \in \boldsymbol{R}^n \ni T(\mathbf{x}) = \mathbf{b}$.

One-to-one is uniqueness question

Projections are neither, others above are both

Example: let T be a linear transformation whose standard matrix is:

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\boldsymbol{R}^4$ onto $\boldsymbol{R}^3$? Is T one-to-one?

Solution:

1) A in echelon form, and pivot in every row $\therefore A\mathbf{x} = \mathbf{b}$ is always consistent $\Rightarrow$ onto
2) $\exists$ free variable $\therefore$ each $\mathbf{b}$ is image of infinite $\mathbf{x}$s $\Rightarrow$ *not* one-to-one

Theorem: $T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation is one-to-one $\Leftrightarrow T(x) = \mathbf{0}$ has only trivial solution

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation whose standard matrix is A. Then:

1) T onto $\mathbb{R}^m \Leftrightarrow$ col(A) span $\mathbb{R}^m$

2) T one-to-one $\Leftrightarrow$ col(A) linearly independent

Example: $T(x_1, x_2) = 3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2$: one-to-one? Onto $\mathbb{R}^3$?

Solution:

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1) columns not multiples $\Rightarrow$ linearly independent and one-to-one

2) col(A) span $\mathbb{R}^3 \Leftrightarrow \exists$ 3 pivots $\Rightarrow$ not onto $\mathbb{R}^3$