1.10 Linear Models in Business, Science and Engineering

Linear models:

1) *proportional* dependence on individual components
2) individual components are *summed*
3) often arise in natural phenomena
4) closely approximate natural phenomena (with reasonably bounded variables)
5) easily adapted to computers

Applications:

1) nutrition (linear programming)
2) electrical networks
3) difference equations (for dynamic processes)

Constructing a Nutritious Weight-Loss Diet

Popular Cambridge Diet of the 1980s

Very low-calorie powdered formula combining precise balance of nutrients

Problem:

1) nonfat milk: $\gg$ protein $\ddot\smile$; $\gg$ calcium $\ddot\frown$
2) soy flour: $\ll$ calcium $\ddot\smile$; $\gg$ fat $\ddot\frown$
3) whey: $\ll$ fat $\ddot\smile$; $\gg$ carbohydrate $\ddot\frown$
4) etc.

	Nutrients per 100g of			
Nutrient	Nonfat milk	Soy flour	Whey	Diet Supplies in One Day
Protein	36	51	13	33
Carbohydrate	52	34	74	45
Fat	0	7	1.1	3

Problem: find proper combination of ingredients to provide exact amounts of nutrients

Solution: define x_1, x_2 and x_3 to be amounts of ingredients

Approach I: each nutrient separate:

$$\left\{\begin{array}{c} x_1 \text{ units of} \\ \text{nonfat milk} \end{array}\right\} \cdot \left\{\begin{array}{c} \text{protein per unit} \\ \text{of nonfat milk} \end{array}\right\}$$

... similar for soy flour and whey ... similar for carbohydrate and fat

Approach II: more efficient: “nutrient vector” for each ingredient → just one vector equation

$$\left\{\begin{array}{c} x_1 \text{ units of} \\ \text{nonfat milk} \end{array}\right\} \cdot \left\{\begin{array}{c} \text{nutrients per unit} \\ \text{of nonfat milk} \end{array}\right\} = x_1\mathbf{a}_1$$

We have:

1) $\mathbf{a}_1$ is the first column of table above
2) $\mathbf{a}_2$ for soy flour
3) $\mathbf{a}_3$ for whey
4) $\mathbf{b}$ for total nutrient requirements (last column)

In equation form:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

Row reduction of augmented matrix (3 significant digits):

$$\begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .277 \\ 0 & 1 & 0 & .392 \\ 0 & 0 & 1 & .233 \end{bmatrix}$$

Diet: 27.7 g nonfat milk, 39.2 g soy flour and 23.3 g whey — YUM!

Note: feasibility requirement: $x_i \geq 0$ ("linear programming") $\rightarrow$ can require large number of ingredients

Cambridge Diet: 31 nutrients, 33 ingredients

Linear Equations and Electrical Networks

Voltage source, e.g., battery: forces current of electrons to flow

Voltage drop through resistor, e.g., lightbulb or motor: Ohm's law:

$$V = RI$$

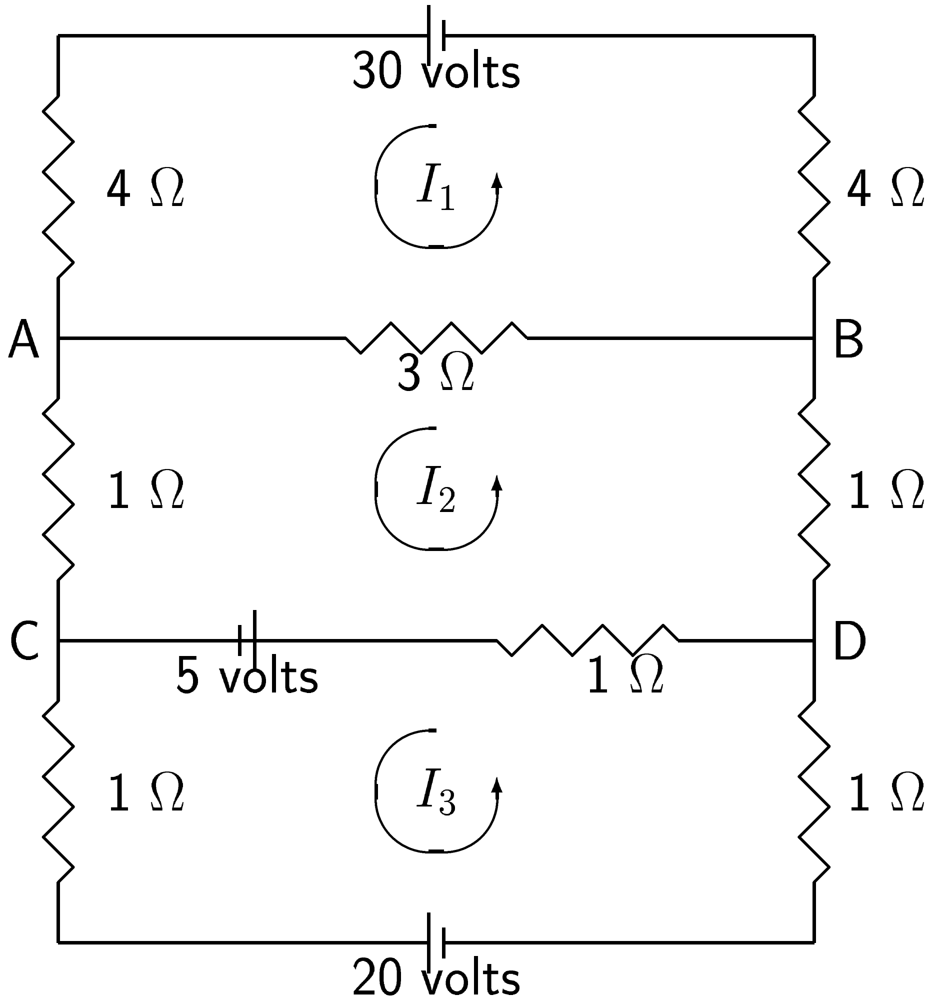
where V = voltage (*volts*), R = resistance (*ohms*, Ω), I = current flow (*amperes* or *amps*)

Note:

1) current direction chosen arbitrarily
2) negative current means flow in opposite direction
3) current direction from longer (positive) side of battery, around to negative $\rightarrow$ voltage is positive

Kirchoff's Voltage Law: sum of voltage drops around loop equals sum of voltage sources (in same direction)

Problem: Determine the loop currents in the network.



Loop 1: I_1 flows through 3 resistors, total $R\,I$ voltage drop:

$$4I_1 + 4I_1 + 3I_1 = 11I_1$$

I_2 also flows through Loop 1 in *opposite* direction: $-3I_2$

Voltage source in Loop 1: +30 volts, therefore:

$$11I_1 - 3I_2 = 30$$

Loop 2:

$$-3I_1 + 6I_2 - I_3 = 5$$

Loop 3:

$$-I_2 + 3I_3 = -25$$

Altogether:

$$\begin{array}{rcrcrcr} 11I_1 & - & 3I_2 & & & = & 30 \\ -3I_1 & + & 6I_2 & - & I_3 & = & 5 \\ & - & I_2 & + & 3I_3 & = & -25 \end{array}$$

Leads to: $I_1 = 3$ amps, $I_2 = 1$ amp and $I_3 =$ -8 amps (i.e., clockwise).

Note:

1) Ohm's law (proportionality) and Kirchoff's law (summability) provide linearity
2) Superposition: solution equal sum of solutions for individual voltage sources
3) Kirchoff's current law: branch current equal sum of loop currents (accounting for loop directions):
   a) D→B $= I_2 = 1$ amp
   b) A→B $= I_1 - I_2 = 2$ amps

c) C→D $= I_2 - I_3 = 9$ amps

Difference Equations

Ecology, economics, engineering, etc.: dynamic systems changing over time

Each "state vector": $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, contains n measured system features at time of kth measurement

Recurrence relation: if $\exists\ A \ni \mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1$, or generally:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad k = 0, 1, 2, \ldots$$

Later: description of behavior as $k \to \infty$

Population migration:

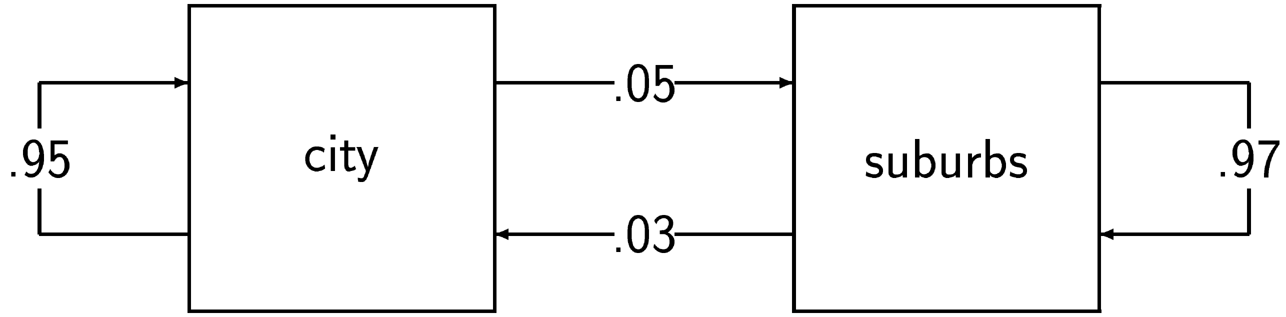
$$\mathbf{x}_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} \quad \begin{array}{l} \text{city population, 2000} \\ \text{suburban population, 2000} \end{array}$$

For 2001, 2002, 2003, etc., we have

$$\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} r_3 \\ s_3 \end{bmatrix}$$

Goal: relate vectors to each other.

Assume annual migration:



After one year, r_0 distributed as:

$$\begin{bmatrix} .95r_0 \\ .05r_0 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} \quad \begin{matrix} \text{remain in city} \\ \text{move to suburbs} \end{matrix}$$

Similarly, s_0 distributed as:

$$s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} \quad \begin{matrix} \text{move to city} \\ \text{remain in suburbs} \end{matrix}$$

These account for 2001 population (assuming closed system):

$$\begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

In matrix form we have:

$$\mathbf{x}_1 = M\mathbf{x}_0$$

with migration matrix M:

$$\begin{array}{cccl} & \multicolumn{2}{c}{\text{from:}} & \\ & \text{city} & \text{suburbs} & \text{to:} \\ \left[\begin{array}{c} \\ \\ \end{array}\right. & \begin{array}{c} .95 \\ .05 \end{array} & \begin{array}{c} .03 \\ .97 \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right] & \begin{array}{l} \text{city} \\ \text{suburbs} \end{array} \end{array}$$

We have the same for $\{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots\}$.

E.g. if in 2000: 600,000 in the city and 400,000 in the suburbs:

$$\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600,000 \\ 400,000 \end{bmatrix} = \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix}$$

Similarly:

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 582,000 \\ 418,000 \end{bmatrix} = \begin{bmatrix} 565,440 \\ 434,560 \end{bmatrix}$$

Note linearity due to:

1) movement and remaining *proportional* to population

2) cumulative effect is *summation* of components

Questions:

1) What is $\mathbf{x}_k$ for large k?

2) What happens as $k \to \infty$?

3) Is this calculatable directly?

4) Does this settle down to an equilibrium?

5) Do cities eventually empty out?

6) Is this true for all such cases?

. . . later . . .